Geometric multigrid methods based on Generic Approximate Sparse Inverses

G.A. Gravvanis

and

C.K. Filelis-Papadopoulos

Τμήμα Ηλεκτρολόγων Μηχανικών και Μηχανικών Υπολογιστών

Department of Electrical and Computer Engineering

ΔΗΜΟΚΡΙΤΕΙΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΘΡΑΚΗΣ

DEMOCRITUS UNIVERSITY OF THRACE

Purpose of this work

- Study the applicability of the multigrid methods in conjunction with Generic Approximate Sparse Inverses.
- Study the convergence behavior and performance for solving problems discretized with higher order discretization schemes.
- Compare the performance and convergence behavior of the proposed schemes for various values of the levels of fill and various orders for three-dimensional model problems.

Introduction

- Let us consider the following PDE:

$$\sum_{i,j=1}^{N=2} \frac{\partial}{\partial x_i}\left[a_{i,j}(x)\frac{\partial u}{\partial x_j}\right] + \sum_{j=1}^{N=2}\left[b_j(x)\frac{\partial u}{\partial x_j}\right] + c(x)u = f, \;\; x \in \Omega$$

- Subject to general boundary conditions:

$$\alpha(x)u + \beta(x)\partial u/\partial \eta = \gamma(x), \quad x \in \partial\Omega$$

- Discretizing the 2D PDE with the Mehrstellen FD scheme or the 3D PDE with the seven point stencil

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} u = 6h^2 f + \frac{1}{2}h^4 \nabla^2 f \qquad \begin{bmatrix} & & \\ & -1 & \\ & & \end{bmatrix}\begin{bmatrix} & -1 & \\ -1 & 6 & -1 \\ & -1 & \end{bmatrix}\begin{bmatrix} & & \\ & -1 & \\ & & \end{bmatrix} u = h^2 f$$

Introduction

- The following linear system is derived

$$Au=f$$

- Multigrid methods can be used to solve the above systems
- Multigrid methods:
  - Near optimal complexity of order O(n)
  - h-elliptical convergence for various problems
  - Consisted of 4 basic components

Mutligrid Methods

- The 4 components that Multigrid methods require are:
  - Smoother or stationary iterative method

$$u^{(k+1)} = u^{(k)} + \omega M\left(f - Au^{(k)}\right), k = 0,1,\ldots$$

  - Restriction operator

$$R : A_{\ell} \to A_{\ell-1}$$

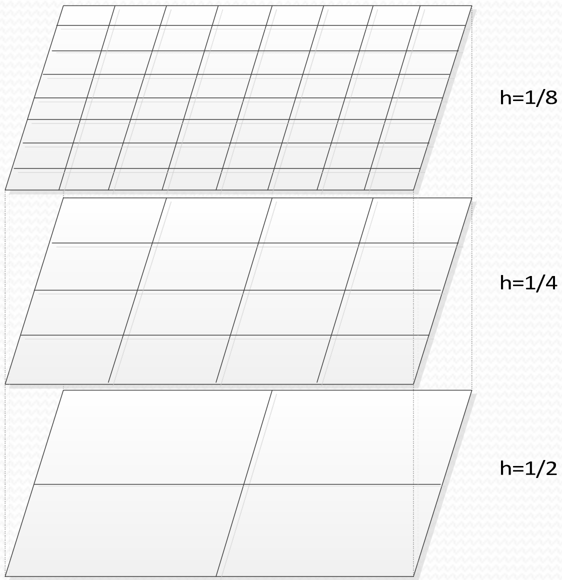
  - Prolongation operator

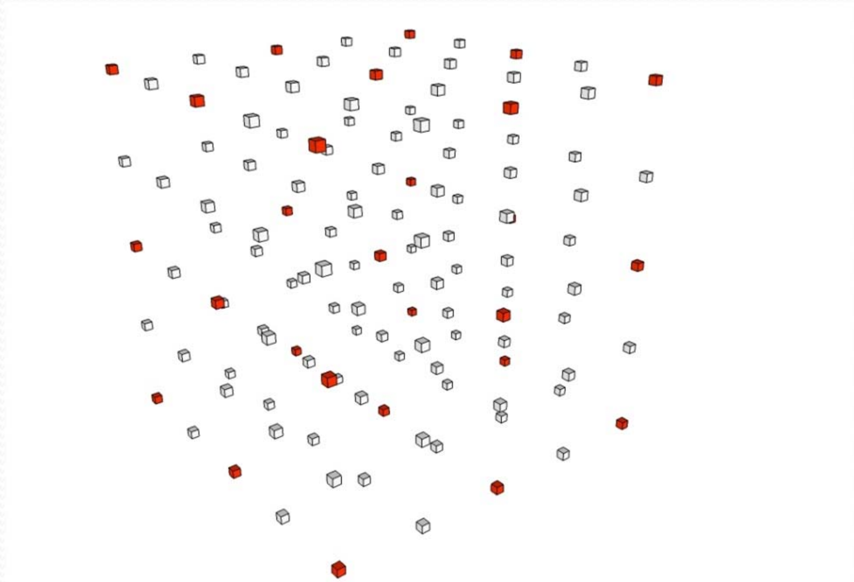
$$P : A_{\ell-1} \to A_{\ell}$$

  - Cycle Strategy (The sequence in which grids are visited and the respective corrections are obtained)
- Multigrid method requires a grid hierarchy

Grid Hierarchy

- 2D and 3D grid hierarchy





- Grid hierarchy can be constructed using Galerkin condition $A_{\ell-1} = RA_{\ell}P$

Transfer Operators

- Transfer operators are used to transfer vectors from coarser to finer and finer to coarser grids.
- A standard choice for the Prolongation operator is linear interpolation. For 2D and 3D discretizations the stencils are as follows:

$$P = \frac{1}{4}\left]\begin{matrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 1\end{matrix}\right[_{2h}^{h} \qquad P = \frac{1}{16}\left]\left]\begin{matrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 1\end{matrix}\right[_{2h}^{h}\left]\begin{matrix}2 & 4 & 2\\ 4 & 8 & 4\\ 2 & 4 & 2\end{matrix}\right[_{2h}^{h}\left]\begin{matrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 1\end{matrix}\right[_{2h}^{h}\right[$$

- The Restriction operator can be defined as $R = cP^{T}, c \in R^{*}$.
- Transfer operators related through the Galerkin conditions simplify mapping on the data.

Smoothers

- Smoothers are stationary iterative methods. Let as consider the Explicit Preconditioned Richardson's method: $u^{(k+1)} = u^{(k)} + \omega M\left(f - Au^{(k)}\right), k = 0,1,\ldots$
- Where M is a preconditioner.
- Explicit preconditioned iterative methods have been extensively used to solve various classes of problems.
- Preconditioning methods are split into two categories: Implicit and Explicit.
- Implicit Preconditioning such as ILU are inherently serial and perform poorly on parallel systems.

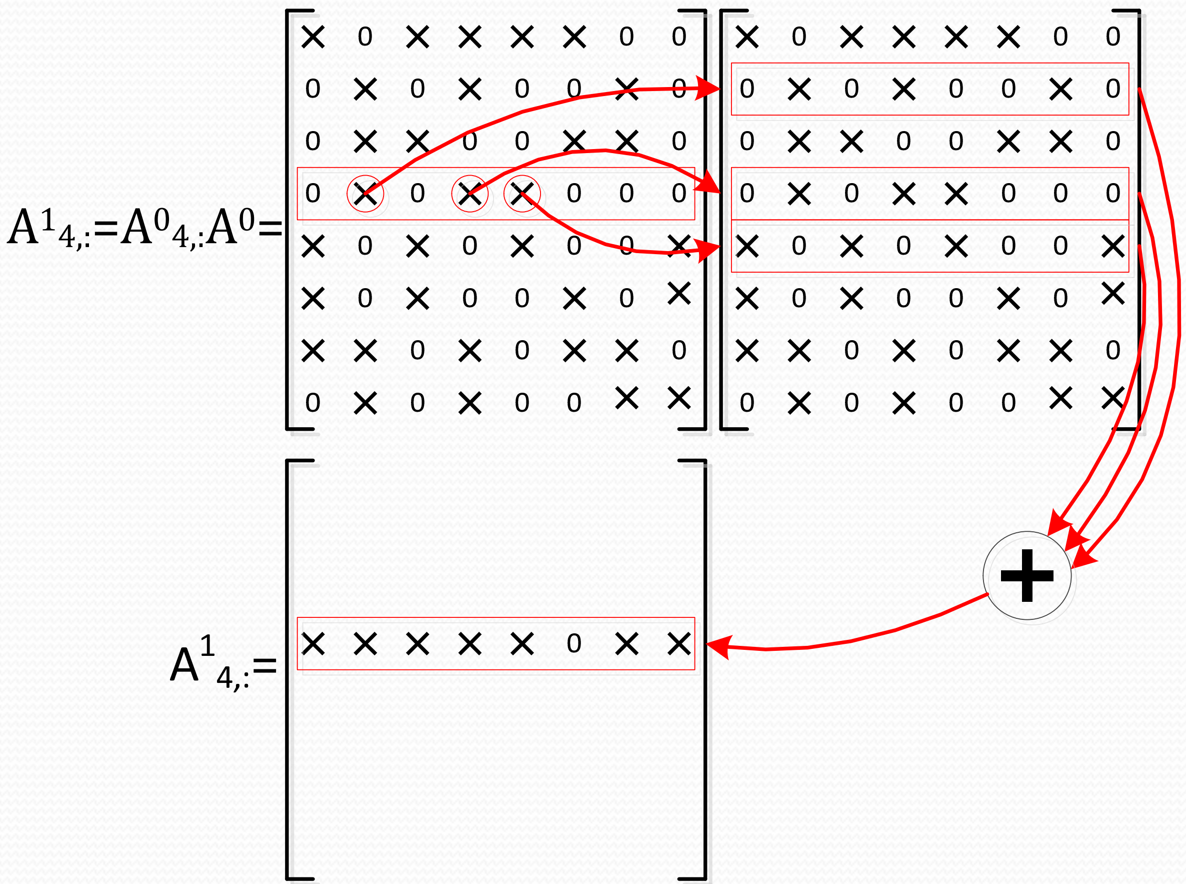
Approximate Inverse

- Various Approximate Inverses have been proposed, based on the minimization of the Frobenius norm of the error, such as SPAI.
- The effectiveness of Explicit Approximate Inverses (EAI) relies on the fact that are close approximants to inverse of a matrix.
- Recently, a Generic Banded (GenAbI) class was proposed that can handle any sparsity of the coefficient matrix and has been used in conjunction with the algebraic multigrid.

Generic Approximate Sparse Inverse (GenAspI)

- The GenAspI is based on a modified approach of GenAbI in conjunction with Approximate Inverse Sparsity Patterns.
- The sparsity pattern is computed by:
  - Sparsifying matrix A against a given drop tolerance (drptol)
  - Finding the lfill levels of neighbors of i-th vertex of the graph of the sparsified matrix A and use them as pattern for the i-th row of the approximate inverse M

Generic Approximate Sparse Inverse (GenAspI)



Generic Approximate Sparse Inverse (GenAspI)

- After the computation of a sparsity pattern the inverse is computed only for the elements dictated by the a priori known sparsity pattern.
- Let us consider the ILU factorization of a matrix A:

$$A=LU+E$$

- where E is the error matrix. The GenAspI can be computed as follows:

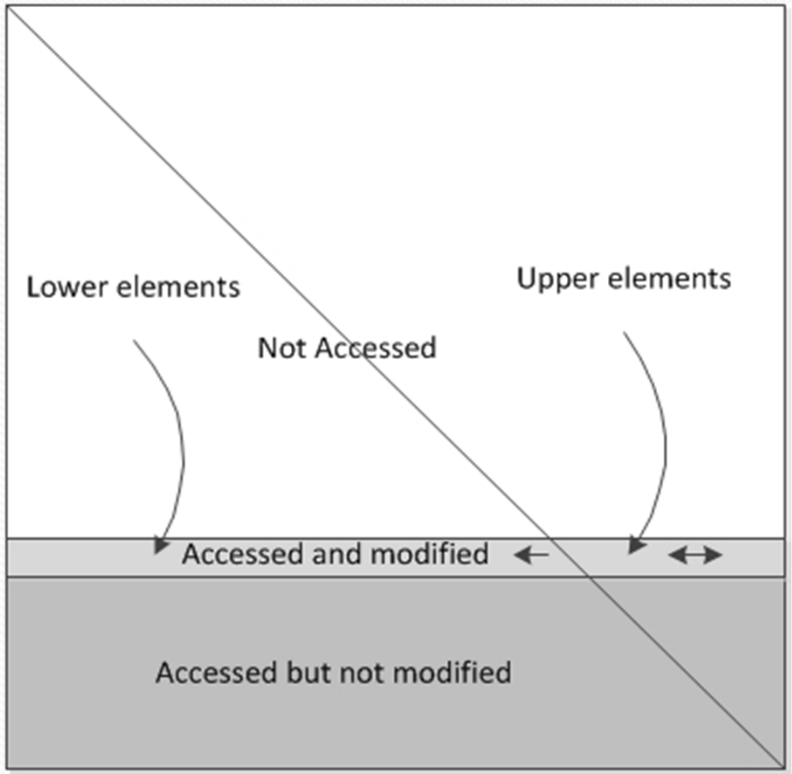
$$UM_{drptol}^{lfill}=I \text{ and } LM_{drptol}^{lfill}=0$$

- The elements of GenAspI are computed by solving the above systems

Generic Approximate Sparse Inverse (GenAspI)

- The elements are computed as follows:

$$\mu_{ij} = \begin{cases} \left(1 - \sum_{k=i+1}^{n} u_{ik}\mu_{ki}\right) \Big/ u_{ii} , & i = j \\ -\sum_{k=j+1}^{n} \mu_{ik}\ell_{kj} , & i > j \\ \left(-\sum_{k=i+1}^{n} u_{ik}\mu_{kj}\right) \Big/ u_{ii} , & i < j \end{cases}$$



Generic Approximate Sparse Inverse (GenAspI)

- It should be noted that GenAspI in conjunction with Explicit Preconditioned Bi-Conjugate Gradient STABilized method had been proven to possess better convergence behavior than SPAI in various model problems from various scientific fields (CFD, CSA, …).
- Moreover, Explicit Approximate Inverses have been proven to be better smoothers than Jacobi or Gauss-Seidel method.
- Furthermore, GenAspI method possesses better convergence behavior the GenAbI or EAI as well as other Explicit Approximate Inverses i.e OBGAIM, etc.

Generic Approximate Sparse Inverse (GenAspI)

- The complexity for computing the Generic Approximate Sparse Inverse (GenAspI) is
  $$(3/4)\left(\mathrm{nnz}(M)^2/n\right)-(3/8)(\mathrm{nnz}(M))+(5/8)n$$
  multiplications and
  $$(3/4)\left(\mathrm{nnz}(M)^2/n\right)-(3/2)(\mathrm{nnz}(M))+(3/4)n$$
  additions.
- In order for a smoother to be effective must satisfy the smoothing property. The smoothing property has been proven for the Optimized Banded Generalized Approximate Inverse (OBGAIM) matrix.Equivalently, the smoothing property can be proven for the GenAspI matrix.

Generic Approximate Sparse Inverse (GenAspI)

- The smoothing property has been proven by Hackbusch for various elliptic PDEs.
- Sharp estimates have been proven for various smoothers or roughers (Bank and Douglas).
- The general form of a smoother, based on the GenAspI is the following:

$$x_l^{(k+1)} = x_l^{(k)} + \omega\left(S\left(x_l^{(k)}\right) - x_l^{(k)}\right) \quad S\left(x_l^{(k)}\right) = x_l^{(k)} + \left(M_{drptol}^{lfill}\right)_l\left(f_l - A_l x_l^{(k)}\right)$$

- The damping parameter ω is affecting the effectiveness of the smoothing scheme.
- The computation of the optimal value for ω is not trivial. Thus, a dynamic method should be used.

Generic Approximate Sparse Inverse (GenAspI)

- To further improve the convergence behavior of the proposed multigrid schemes, the Dynamic Over/Under Relaxation scheme is used.

$$x_\ell^{(k+1)} = x_\ell^{(k)} + \omega_e M\left(b_\ell - Ax_\ell^{(k)}\right), \omega_e = \omega(1+\kappa)$$

$$\kappa = \frac{\left\langle \Delta x_\ell^{(k)}, b_\ell - A_\ell\left(x_\ell^{(k)} + \omega M_{\text{drptol}}^{\text{lfill}}\left(b_\ell - A_\ell x_\ell^{(k)}\right)\right)\right\rangle}{\left\langle \Delta x_\ell^{(k)}, A_\ell \Delta x_\ell^{(k)}\right\rangle}$$

- Two-stage non-stationary smoother in order to automatically (dynamically) compute the optimal value of effective damping parameter ω_e.

Restriction operator

- The restriction (projection) operator is used to transfer vectors from a finer to a coarser grid.
- An effective choice for the restriction operator is the full-weighting (2D and 3D):

$$R = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_h^{2h} \qquad R = \frac{1}{64}\left[\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_h^{2h}\begin{bmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 2 \end{bmatrix}_h^{2h}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_h^{2h}\right]$$

- It can be observed that both for the two-dimensional and three-dimensional case the elements of the coarse vector are the weighted average of their neighbors in the finer grid.

Prolongation Operator

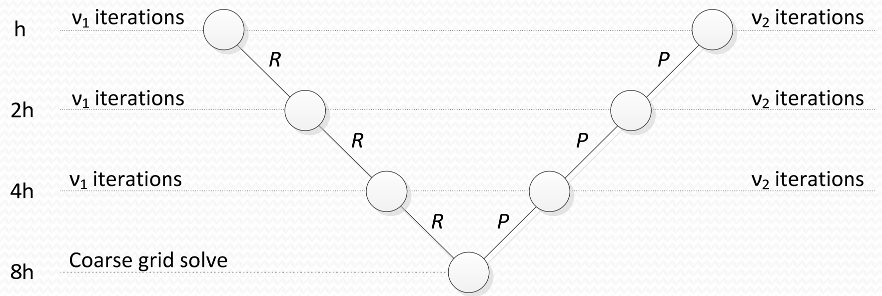
- The Prolongation (Interpolation) operator is used to transfer vectors from a coarse to a fine grid.
- An effective choice for the prolongation operator is the bilinear (trilinear) interpolation for 2D (3D) problems:

$$P = \frac{1}{4}\left]\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix}\right[_{2h}^{h} \qquad P = \frac{1}{16}\left]\left]\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix}\right[_{2h}^{h} \left]\begin{matrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 2 \end{matrix}\right[_{2h}^{h} \left]\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix}\right[_{2h}^{h}\right[$$

- Full-weighting operator and the bilinear interpolation are related through the Galerkin condition $P=R^T$.
- Transfer operators occupy about 30% of the total computational work of the multigrid algorithm, thus choosing higher order interpolation schemes may lead to excessive computational cost at no extra gain in the convergence behavior

Cycle Strategy

- Cycle strategy refers to the sequence in which the grids are visited and the respective corrections are obtained.
- The most popular choice is the so-called V-Cycle:



- The method descends to coarser level executing ν_1 smoother iterations in each level and then the method ascends executing ν_2 iterations in each level

Numerical Results

- The numerical tests were performed on a Dual Socket AMD Opteron Processor 6128 HE, with 16GB RAM, running Ubuntu Linux 12.04.1.
- **Model Problem I:** Let us consider a PDE in two space variables as follows:

$$-\Delta u = 2\left[\left(1-6x^2\right)y^2\left(1-y^2\right)+\left(1-6y^2\right)x^2\left(1-x^2\right)\right],\ (x,y)\in\Omega\equiv[0,1]\times[0,1]$$

- $u(x,y)=0, \quad (x,y)\in\Omega$
- where Ω is the unit square, is the boundary of the domain and has been discretized with the fourth order Mehrstellen scheme.

Numerical Results

- **Model Problem II:** Let us consider a PDE in three space variables as follows:

$$-\Delta u = 1, (x, y, z) \in \Omega \equiv [0,1]\times[0,1]\times[0,1]$$

- $u(x,y,z)=0, \quad (x,y,z)\in\Omega$
- where Ω is the unit cube, is the boundary of the domain and has been discretized with the seven point stencil.
- The termination criterion was set to $||r_i||<1e\text{-}10||r_0||$. The pre-smoothing and post-smoothing steps were set to $\nu_1=2$ and $\nu_2=1$. The drop tolerance for the computation of the GenAspI matrices was set to drptol=0.0.

Model Problem I

GenAspI Performance

n	lfill=1	lfill=2	lfill=3	lfill=4
65025	0.726184	2.142500	4.919570	9.839040
261121	2.960540	8.849770	20.320700	40.508400
1046529	12.130800	35.992600	82.929300	166.288000

GenAspI-Multigrid Performance

n	lfill=1	lfill=2	lfill=3	lfill=4
65025	0.906318	0.819446	0.579109	0.750173
261121	4.149710	3.866150	2.891410	3.061560
1046529	19.014800	17.339700	12.962000	15.875600

GenAspI-Multigrid convergence behavior and convergence factors

n	lfill=1	lfill=2	lfill=3	lfill=4
65025	16/0.216	11/0.123	6/0.020	6/0.017
261121	16/0.221	12/0.125	7/0.023	6/0.019
1046529	16/0.227	12/0.129	7/0.027	7/0.0271

Model Problem II

GenAspI Performance

n	lfill=1	lfill=2	lfill=3	lfill=4
29791	0.241650	1.013130	3.293560	8.917160
250047	2.139860	9.285240	30.929700	86.065600
2048383	18.547500	80.562700	279.175000	798.750000

GenAspI-Multigrid Performance

n	lfill=1	lfill=2	lfill=3	lfill=4
29791	0.179297	0.204390	0.318714	0.475071
250047	1.712660	2.298260	3.904100	6.214280
2048383	15.655400	21.998500	38.490000	60.606200

GenAspI-Multigrid convergence behavior and convergence factors

n	lfill=1	lfill=2	lfill=3	lfill=4
29791	9/0.062	7/0.034	7/0.032	7/0.030
250047	9/0.061	8/0.041	8/0.039	8/0.039
2048383	9/0.062	8/0.043	8/0.041	8/0.040

Conclusion

- It should be noted that the proposed multigrid scheme based on GenAspI has better convergence behavior compared to multigrid methods based on Generic Approximate Banded Inverse (GenAbI) or Explicit Approximate Inverses (EAI).
- It should be also noted that the convergence behavior of the multigrid method based on various families of Approximate Inverses is better than classical smoothers such as Jacobi and Gauss-Seidel.
- Finally, research is focused in improving further the performance of the GenAspI in conjunction with the sparsity pattern. Additionally, Generic and Explicit Approximate Inverses have been experimentally shown to be better, in terms of performance and convergence behavior, in comparison with SPAI and FSAI schemes.

References

- Axelsson, O. 1996. *Iterative solution methods.* Cambridge University Press.
- Bank, R.E., and Douglas, C.C., 1985. Sharp estimates for multigrid rates of convergence with general smoothing and acceleration, *SIAM J.Num. Anal.* 22, 617-633.
- Brandt, A. 1977. Multi-level adaptive solutions to boundary-value problems. *Math. Comp*. 31, 333-390.
- Bröker, O., Grote, M.J., Mayer, C., and Reusken, A. 2001. Robust parallel smoothing for multigrid via sparse approximate inverses. *SIAM J. Sci. Comput.* 23, 4, 1396–1417.
- Chow, E. 2000. A priori sparsity patterns for parallel sparse approximate inverse preconditioners. *SIAM J. Sci. Comput.* 21, 1804-1822.
- Filelis-Papadopoulos, C.K., and Gravvanis, G.A. 2014. Generic approximate sparse inverse matrix techniques, *Int. J. Comput. Meth.*, in press.
- Filelis-Papadopoulos, C.K., and Gravvanis, G.A. 2013. On the multigrid method based on finite difference approximate inverses, *CMES* 90, 3, 233-253.
- Gravvanis, G.A., Filelis-Papadopoulos, C.K., and Matskanidis, P.I. Algebraic multigrid methods based on Generic Approximate Matrix Techniques, TR/ECE/ASC-AMA/2012/13, submitted.
- Gravvanis, G.A. 2002. Explicit approximate inverse preconditioning techniques. *ARCME* 9, 4, 371-402.
- Hackbusch, W. 1982. Multi-grid convergence theory. In *Lecture Notes in Mathematics* 960, Springer,177-219.
- Haelterman, R., Vierendeels, J., and Van Heule, D. 2006. Non–stationary two–stage relaxation based on the principle of aggregation multi-grid. In *Computational Fluid Dynamics 2006*, Part 3, Springer, 243–248.
- Trottenberg, U., Osterlee, C. W., and Schuller, A. 2000. *Multigrid*, Academic Press.

Thank you for your attention